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Exact solutions of the Schrödinger equation with non-central potential by the Nikiforov–Uvarov method

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Abstract

The general solutions of the Schrödinger equation for a non-central potential are obtained by using the Nikiforov–Uvarov method. The Schrödinger equation with general non-central potential is separated into radial and angular parts, and energy eigenvalues and eigenfunctions are derived analytically. By making special selections, the non-central potential is reduced to Coulomb and Hartmann ring-shaped potentials, and the obtained results are compared with the solutions of Coulomb and Hartmann potentials given in the literature.

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1. Introduction

One of the interesting problems of non-relativistic quantum mechanics is to find exact solutions of the Schrödinger equation for certain potentials of physical interest. Exact solutions of this equation are possible only for certain potentials such as Coulomb, Morse, Pöschl–Teller, Hulthen and harmonic oscillator, etc [1]. Other exactly solvable ones are the ring-shaped potentials introduced by Hartmann [2] and Quesne [3]. These potentials involve an attractive Coulomb potential with a repulsive inverse square potential.

In recent years, considerable efforts have been made to obtain the analytical solution of non-central problems. In particular, the Coulombic ring-shaped potential [4] revived in quantum chemistry by Hartmann and co-workers [5] and the oscillatory ring-shaped potential, systematically studied by Quesne [3], have been investigated from a quantum mechanical viewpoint by using various approaches.

The Coulombic ring-shaped, or Hartmann, potential is

$$V = -Z\frac{1}{\sqrt{x_1^2 + x_2^2 + x_3^2}} + \frac{1}{2}Q\frac{1}{x_1^2 + x_2^2}, \qquad Z > 0, \quad Q > 0, \tag{1}$$

where $Z = \eta\sigma^2$ and $Q = q\eta^2\sigma^2$ in the notation of Hartmann and of Kibler and Negadi [4]. This potential in the limiting case $Q = 0$ reduces to an attractive Coulomb potential and a special case of the potential (in spherical coordinates)

$$V(r,\theta) = \frac{\alpha}{r} + \frac{\beta}{r^2\sin^2\theta} + \gamma\frac{\cos\theta}{r^2\sin^2\theta} \tag{2}$$

introduced by Makarov *et al* [6]. This potential can be used in quantum chemistry and nuclear physics to describe ring-shaped molecules such as benzene and interactions between deformed pairs of nuclei. There are different methods used to obtain exact solutions of the Schrödinger equation for a non-central potential which are supersymmetric (SUSYQM) [7, 8], path integral representation [9, 10], Bessel [11] and dynamical (or non-invariance) group [12].

In this paper, we introduce an alternative, elegant and simple method for an algebraic solution of the Schrödinger equation with non-central potential. This method is called the Nikiforov–Uvarov (NU) method [13] and is based on solving the second-order linear differential equations by reducing to a generalized equation of hypergeometric type.

The NU method is used to solve Schrödinger, Dirac, Klein–Gordon and Duffin–Kemmer–Petiau wave equations in the presence of exponential-type potentials such as Woods–Saxon [14], Pöschl–Teller [15] and Hulthen [16, 17]. The aim of this study is to show that the Nikiforov–Uvarov method can be used to obtain exact solutions of non-central potentials. Thus, radial and angular parts of the Schrödinger equation with non-central potential are solved by the NU method and it is seen that this method is applicable not only to exponential but also to non-central-type potentials.

This paper is arranged as follows: in section 2, the Schrödinger equation in spherical coordinates for a particle in the presence of a non-central potential is separated into radial and angular parts. In section 3, the Nikiforov–Uvarov method is given briefly. Then in section 4, the solutions of the radial and angular parts of the Schrödinger equation and the special cases of non-central, i.e., Hartmann and Coulomb potential, solutions are obtained and compared with studies using different methods in the literature. Finally, the relevant results are discussed in section 5.

2. Separating variables of the Schrödinger equation with non-central potential

In spherical polar coordinates, the Schrödinger equation for a particle in the presence of a non-central potential $V(r,\theta)$ becomes

$$-\frac{\hbar^2}{2m}\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right]\psi(r,\theta,\varphi) + \left(\frac{\alpha}{r} + \frac{\beta}{r^2\sin^2\theta} + \frac{\gamma\cos\theta}{r^2\sin^2\theta}\right)\psi(r,\theta,\varphi) = E\psi(r,\theta,\varphi). \tag{3}$$

If we assign the corresponding spherical total wavefunction as $\psi(r,\theta,\varphi) = R(r)Y(\theta,\varphi)$, we get

$$\frac{1}{R(r)}\frac{\partial}{\partial r}\left(r^2\frac{\partial R(r)}{\partial r}\right) - \frac{2m\alpha r}{\hbar^2} + \frac{2mEr^2}{\hbar^2} + \frac{1}{Y(\theta,\varphi)}\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y(\theta,\varphi)}{\partial\theta}\right) - \frac{2m}{\hbar^2}\left(\frac{\beta}{\sin^2\theta} + \frac{\gamma\cos\theta}{\sin^2\theta}\right) + \frac{1}{Y(\theta,\varphi)}\frac{1}{\sin^2\theta}\frac{\partial^2 Y(\theta,\varphi)}{\partial\varphi^2} = 0. \tag{4}$$

Separating the Schrödinger equation into variables and selecting $Y(\theta,\varphi) = \Theta(\theta)\Phi(\varphi)$, the following equations are obtained:

$$\frac{\mathrm{d}^2R(r)}{\mathrm{d}r^2} + \frac{2}{r}\frac{\mathrm{d}R(r)}{\mathrm{d}r} + \frac{2mr^2}{\hbar^2}\left(E - \frac{\alpha}{r} - \frac{\lambda}{r^2}\right)R(r) = 0, \tag{5a}$$

$$\frac{\mathrm{d}^2\Theta(\theta)}{\mathrm{d}\theta^2} + \cot\theta\frac{\mathrm{d}\Theta(\theta)}{\mathrm{d}\theta} + \left(\lambda - \frac{m^2}{\sin^2\theta} - \frac{2m}{\hbar^2}\left(\frac{\beta+\gamma\cos\theta}{\sin^2\theta}\right)\right)\Theta(\theta) = 0, \tag{5b}$$

$$\frac{\mathrm{d}^2\Phi(\varphi)}{\mathrm{d}\varphi^2} + m^2\Phi(\varphi) = 0, \tag{5c}$$

where m^2 and λ are separation constants. It is well known that the azimuthal angle solution of equation (5*c*) is

$$\Phi_m = A\,\mathrm{e}^{\mathrm{i}m\varphi} \qquad (m = 0, \pm1, \pm2, \ldots).$$

Equations (5*a*) and (5*b*) are radial and polar angle equations, and they will be solved by using the Nikiforov–Uvarov method [13], given briefly in the following section.

3. Nikiforov–Uvarov method

In this method, for a given real or complex potential, the Schrödinger equation in one dimension is reduced to a generalized equation of hypergeometric type with an appropriate coordinate transformation $s = s(r)$ and it can be written in the following form:

$$\psi(s)'' + \frac{\tilde{\tau}(s)}{\sigma(s)}\psi'(s) + \frac{\tilde{\sigma}(s)}{\sigma^2(s)}\psi(s) = 0, \tag{6}$$

where $\sigma(s)$ and $\tilde{\sigma}(s)$ are polynomials, at most second degree, and $\tilde{\tau}(s)$ is a first-degree polynomial. Hence, from equation (6), the Schrödinger equation and the Schrödinger-like equations can be solved by means of the special potentials for some quantum mechanics problems. To find the particular solution of equation (6) by separation of variables, if one deals with the transformation

$$\psi(s) = \phi(s)y(s), \tag{7}$$

it reduces to an equation of hypergeometric type

$$\sigma(s)y'' + \tau(s)y' + \grave{\lambda}y = 0, \tag{8}$$

and $\phi(s)$ is defined as a logarithmic derivative

$$\phi'(s)/\phi(s) = \pi(s)/\sigma(s). \tag{9}$$

The other part $y(s)$ is the hypergeometric-type function whose polynomial solutions are given by the Rodrigues relation

$$y_n(s) = \frac{B_n}{\rho(s)}\frac{\mathrm{d}^n}{\mathrm{d}s^n}[\sigma^n(s)\rho(s)], \tag{10}$$

where B_n is a normalizing constant and the weight function $\rho(s)$ must satisfy the condition

$$(\sigma\rho)' = \tau\rho. \tag{11}$$

The function π and the parameter $\grave{\lambda}$ required for this method are defined as follows:

$$\pi(s) = \frac{\sigma' - \tilde{\tau}}{2} \pm \sqrt{\left(\frac{\sigma' - \tilde{\tau}}{2}\right)^2 - \tilde{\sigma} + k\sigma}, \tag{12}$$

$$\grave{\lambda} = k + \pi'. \tag{13}$$

On the other hand, in order to find the value of k, the expression under the square root must be the square of the polynomial. Thus, a new eigenvalue equation for the Schrödinger equation becomes

$$\grave{\lambda} = \grave{\lambda}_n = -n\tau' - \frac{n(n-1)}{2}\sigma'', \tag{14}$$

where

$$\tau(s) = \tilde{\tau}(s) + 2\pi(s) \tag{15}$$

and its derivative is negative.

4. Solutions of the radial and polar angle equations with the NU method

4.1. Eigenvalues and eigenfunctions of the radial equation

Assuming that $R(r) = (1/r)F(r)$ is bounded as $r \to 0$, the radial Schrödinger equation given in equation (5*a*) is

$$F''(r) + \left(\frac{2mE}{\hbar^2} - \frac{2m\alpha}{\hbar^2 r} - \frac{\ell(\ell+1)}{r^2}\right) F(r) = 0. \tag{16}$$

It is obvious that equation (16) is the same as the problem of an electron in a Coulomb-like field. Letting

$$\frac{2mE}{\hbar^2} = -\varepsilon^2, \qquad \frac{2m\alpha}{\hbar^2} = b^2, \qquad \lambda = \ell(\ell+1), \qquad \alpha = -Ze^2,$$

and substituting these expressions in equation (16), one obtains

$$F''(r) + (-\varepsilon^2 r^2 - b^2 r - \lambda)\frac{1}{r^2} F(r) = 0. \tag{17}$$

If we apply the NU method by comparing equation (17) with equation (6), the following expressions are obtained:

$$\tilde{\tau} = 0, \qquad \sigma = r, \qquad \tilde{\sigma} = -\varepsilon^2 r^2 - b^2 r - \lambda.$$

Inserting these polynomials in equation (12), we achieve the π function as

$$\pi = \tfrac{1}{2} \pm \tfrac{1}{2}\sqrt{4\varepsilon^2 r^2 + 4r(k + b^2) + 4\lambda + 1}. \tag{18}$$

According to the NU method, the expression in the square root must be the square of the polynomial.

Hence, we can determine that the constant k by using the condition that the discriminant of the square root is zero, that is

$$k = -b^2 \pm 2\sqrt{\varepsilon^2}\left(\ell + \tfrac{1}{2}\right). \tag{19}$$

In view of that, one can find new possible functions for each k as

$$\pi = \begin{cases} \dfrac{1}{2} \pm \left[\sqrt{\varepsilon^2} r + \left(\ell + \dfrac{1}{2}\right)\right], & \text{for} \quad k = -b^2 + 2\sqrt{\varepsilon^2}\left(\ell + \tfrac{1}{2}\right) \\ \dfrac{1}{2} \pm \left[\sqrt{\varepsilon^2} r - \left(\ell + \dfrac{1}{2}\right)\right], & \text{for} \quad k = -b^2 - 2\sqrt{\varepsilon^2}\left(\ell + \tfrac{1}{2}\right). \end{cases} \tag{20}$$

For the polynomial of $\tau = \tilde{\tau} + 2\pi$ which has a negative derivative, we select

$$k = -b^2 - 2\sqrt{\varepsilon^2}\left(\ell + \tfrac{1}{2}\right) \qquad \text{and} \qquad \pi = \tfrac{1}{2} - \left[\sqrt{\varepsilon^2} r - \left(\ell + \tfrac{1}{2}\right)\right].$$

With this selection and $\grave{\lambda} = k + \pi'$, τ and $\grave{\lambda}$ can be written as, respectively,

$$\tau = 2\left(\ell + 1 - \sqrt{\varepsilon^2} r\right), \tag{21}$$

$$\grave{\lambda} = -b^2 - \sqrt{\varepsilon^2}(2\ell + 2). \tag{22}$$

Comparing the definition of $\grave{\lambda}_N$ in equation (14) with the equation

$$\grave{\lambda}_N = 2N\sqrt{\varepsilon^2}, \tag{23}$$

the exact energy eigenvalues of the radial part of the Schrödinger equation with non-central

potential are derived as

$$E_N = -\frac{mZ^2e^4}{\hbar^2}\frac{1}{2(N+\ell+1)^2}, \tag{24}$$

where N denotes the radial quantum number.

Using σ and π in equations (9)–(11), one can find wavefunctions $y(r) = y_{N\ell}(r)$ and $\phi(r)$ from equation (7)

$$F_{N\ell}(z) = C_{N\ell} z^{\ell+1} \exp\left(-\frac{z}{2}\right) L_N^{2\ell+1}(z), \tag{25}$$

where $L_N^{2\ell+1}(z)$ stands for the associated Laguerre functions whose argument is equal to $z = \frac{2\mu Ze^2}{\hbar^2(N+\ell+1)}r$ and $C_{N\ell}$ is the normalization constant determined by $\int_0^\infty F_{N\ell}^2(r)\mathrm{d}r = 1$ [18]. Consequently, the corresponding normalized wavefunctions are found to be

$$F_{n'\ell}(r) = \left(\frac{\mu Ze^2}{\hbar^2 n'}\right)^{1/2}\left(\frac{(n'-\ell-1)!}{n'\Gamma(n'+\ell+1)}\right)^{1/2}\left(\frac{2\mu Ze^2}{\hbar^2 n'}\right)^{\ell+1} r^{\ell+1} \times \exp\left(-\frac{2\mu Ze^2 r}{\hbar^2 n'}\right) L_{n'-\ell-1}^{2\ell+1}\left(\frac{2\mu Ze^2}{\hbar^2 n'}r\right), \tag{26}$$

where $n' = N + \ell + 1$. This equation also stands for the solution of the radial Schrödinger equation with the Coulomb potential, since the radial Schrödinger equation with non-central potential contains only Coulombic potential terms.

4.2. Eigenvalues and eigenfunctions of the polar angle equation

We are now going to derive eigenvalues and eigenfunctions of the polar angle part of the Schrödinger equation with a similar method as given in section 4.1.

Introducing a new variable $x = \cos\theta$, equation (5*b*) is brought to the form

$$\frac{\mathrm{d}^2\Theta(x)}{\mathrm{d}x^2} - \frac{2x}{1-x^2}\frac{\mathrm{d}\Theta(x)}{\mathrm{d}x} + \left(\frac{\lambda(1-x^2) - m^2 - \frac{2m}{\hbar^2}(\beta+\gamma x)}{(1-x^2)^2}\right)\Theta(x) = 0. \tag{27}$$

Comparing with equation (6), the following expressions are obtained: $\tilde{\tau} = -2x$, $\sigma = 1 - x^2$ and $\tilde{\sigma} = -\lambda x^2 - \gamma x + (\lambda - m^2 - \beta)$. Putting them in equation (12), the function π is

$$\pi = \pm\sqrt{x^2(\lambda - k) + \gamma x - (\lambda - m^2 - \beta - k)}.$$

According to the NU method, the expression in the square root must be the square of the polynomial. So, one can find new possible functions for each k as

$$\pi = \pm\begin{cases} x\sqrt{\dfrac{m^2+\beta+u}{2}} + \sqrt{\dfrac{m^2+\beta-u}{2}}, & \text{for} \quad k = \dfrac{2\lambda - m^2 - \beta}{2} - \dfrac{1}{2}u, \\ x\sqrt{\dfrac{m^2+\beta-u}{2}} + \sqrt{\dfrac{m^2+\beta+u}{2}}, & \text{for} \quad k = \dfrac{2\lambda - m^2 - \beta}{2} + \dfrac{1}{2}u, \end{cases} \tag{28}$$

where $u = \sqrt{(m^2+\beta)^2 - \gamma^2}$.

For the polynomial of $\tau = \tilde{\tau} + 2\pi$ which has a negative derivative,

$$\tau = -2\sqrt{\frac{m^2+\beta-u}{2}} - 2x\left(1 + \sqrt{\frac{m^2+\beta+u}{2}}\right), \tag{29}$$

$\dot{\lambda} = k + \pi'$ and another definition $\dot{\lambda}_n = -n\tau' - \frac{n(n-1)}{2}\sigma''$ are given as follows, respectively,

$$\dot{\lambda} = \frac{2\lambda - (m^2+\beta)}{2} - \frac{1}{2}u - \sqrt{\frac{m^2+\beta+u}{2}}, \tag{30}$$

$$\lambda_n = 2n\left(1+\sqrt{\frac{m^2+\beta+u}{2}}\right)+n(n-1). \tag{31}$$

To obtain $E_n = \lambda - (m^2+\beta)$, we compared equations (30) and (31),

$$(2n+1)\sqrt{\frac{m^2+\beta+u}{2}}+\frac{u-(m^2+\beta)}{2}+n(n+1)=\lambda-(m^2+\beta). \tag{32}$$

Using the definition of $\lambda = \ell(\ell+1)$, from equation (32) one obtains

$$\ell = \sqrt{\frac{m^2+\beta+\sqrt{(m^2+\beta)^2-\gamma^2}}{2}}+n. \tag{33}$$

If we substitute equation (33) into eigenvalues of the radial part of the Schrödinger equation with non-central potential, which has a similar radial Schrödinger equation to the Coulomb potential, equation (24), we find that the final energy eigenvalues for a bound electron in a Coulomb potential as well as in a combination of non-central potentials given by equation (5*b*) are

$$E_N = -\frac{mZ^2e^4}{\hbar^2}\frac{1}{2\left(N+\sqrt{\frac{m^2+\beta+\sqrt{(m^2+\beta)^2-\gamma^2}}{2}}+n+1\right)^2}. \tag{34}$$

Then, the wavefunction of the polar angle part of the Schrödinger equation, using σ and π in equations (9)–(11), is obtained:

$$\phi = (1-x)^{B+C/2}(1+x)^{B-C/2}, \tag{35}$$

$$\rho = (1-x^2)^B\left(\frac{1+x}{1-x}\right)^{-C}, \tag{36}$$

$$y_n = B_n(1-x)^{-(B+C)}(1+x)^{-(B-C)}\frac{\mathrm{d}^n}{\mathrm{d}x^n}[(1+x)^{n+B-C}(1-x)^{n+B+C}], \tag{37}$$

where $B=\sqrt{\frac{m^2+\beta+u}{2}}$ and $C=\sqrt{\frac{m^2+\beta-u}{2}}$. The polynomial solution of y_n is expressed in terms of a Jacobi polynomial which is one of the orthogonal polynomials, giving $\approx P_n^{(B+C,B-C)}(x)$. Substituting equations (35) and (37) into equation (7), the corresponding wavefunctions are found to be

$$\Theta_n(x) = N_n(1-x)^{(B+C)/2}(1+x)^{(B-C)/2}P_n^{(B+C,B-C)}(x), \tag{38}$$

where N_n is the normalization constant determined by $\int_{-1}^{+1}[\Theta_n(x)]^2\,\mathrm{d}x = 1$ and using the relation orthogonality of Jacobi polynomials [18, 19], the normalization constant becomes

$$N_n = \sqrt{\frac{(2n+2B+1)\Gamma(n+1)\Gamma(n+2B+1)}{2^{2B+1}\Gamma(n+B+C+1)\Gamma(n+B-C+1)}}. \tag{39}$$

4.3. A special case: Hartmann ring-shaped potential

The ring-shaped Hartmann potential is

$$V_{\mathrm{H}}(r,\theta) = \eta\sigma^2\varepsilon_0\left(\frac{2a}{r}-\frac{q\eta a^2}{r^2\sin^2\theta}\right). \tag{40}$$

Here, $a = \hbar^2/\mu e^2$ (Bohr's radius), $\varepsilon_0 = -\mu e^4/2\hbar^2$ (ground-state energy of the hydrogen atom), η and σ are dimensionless positive parameters which range from about 1 to 10 in theoretical chemistry applications and q is a real parameter. If we compare the Hartmann

potential with the generalized non-central potential given in equation (2), the following terms are obtained: $\alpha = -\eta\sigma^2 e^2$, $\beta = \frac{q\eta^2\sigma^2\hbar^2}{2\mu}$ and $\gamma = 0$. Using the similarity of separated equations for the Hartmann potential to those of the hydrogen atom, one can write immediately from equation (34), the energy spectrum for the Hartmann system is given by

$$E_N = -\frac{\mu(\eta\sigma^2)^2 e^4}{2\hbar^2\left(N + \sqrt{m^2 + \frac{q\eta^2\sigma^2\hbar^2}{2\mu}} + n + 1\right)^2}, \tag{41}$$

and radial wavefunctions of the Hartmann potential are given in equation (26) and are in agreement with [2, 7, 20].

The polar angle wavefunction of the Hartmann potential is obtained by reducing the polar angle wavefunctions of the generalized non-central potential as given in equation (40). When $\gamma = 0$, we simply get the parameters as $u = m^2 + \beta$, $B = (m^2 + \beta)^{1/2}$ and $C = 0$. With the selection of $m' = B$, we obtain

$$\Theta_n(\cos\theta) = \sqrt{\frac{(2n + 2m' + 1)\Gamma(n + 1)\Gamma(n + 2m' + 1)}{2^{2m'+1}\Gamma(n + m' + 1)\Gamma(n + m' + 1)}}(\sin\theta)^{m'} P_n^{(m',m')}(\cos\theta), \tag{42}$$

Here, $P_n^{(m',m')}(\cos\theta)$ is given in terms of ultraspherical polynomials as [18]

$$P_n^{(m',m')}(\cos\theta) = \frac{\Gamma(2m' + 1)}{\Gamma(m' + 1)}\frac{\Gamma(n + m' + 1)}{\Gamma(n + 2m' + 1)} P_n^{(\lambda)}(\cos\theta),$$

and using the definition of ultraspherical polynomials we get equation (42) as follows:

$$\Theta_{\ell m'}(\cos\theta) = \sqrt{\frac{(2n + 2m' + 1)n!}{2\Gamma(n + 2m' + 1)}}(\sin\theta)^{m'} \sum_{\nu=0}^{\left[\frac{\ell-m'}{2}\right]} \frac{(-1)^\nu \Gamma(2\ell - 2\nu + 1)}{2^\ell \nu!(\ell - m' - 2\nu)!\Gamma(\ell - \nu + 1)} \times (\cos\theta)^{(\ell-m'-2\nu)}, \tag{43}$$

where $\ell = n + m'$, $n = 0, 1, 2, \ldots$. Therefore, we derived the polar angle wavefunctions of the Hartmann potential by reducing the polar angle wavefunctions of the generalized non-central potential, and equation (43) is consistent with [2].

5. Conclusions

We have obtained the exact solutions of the radial and angular parts of the Schrödinger equation for non-central potential using the Nikiforov–Uvarov method. This method is usually used in solving analytically Schrödinger, Dirac, Klein–Gordon and Duffin–Kemmer–Petiau wave equations in the presence of exponential-type potentials. In this study, we applied the NU method to non-central potentials by separating the Schrödinger equation into radial and spherical polar coordinates, so we generalized the feasibility of the NU method. Within the framework of this analytical treatment, the radial and polar Schrödinger equation has been transformed to a generalized equation of hypergeometric type, and we have clarified that the Schrödinger equation with respect to confluent hypergeometric polynomials can be exactly solved. As far as we know, this methodology has not been perceived until now for this potential shape. Energy eigenvalues are obtained in a Coulomb potential as well as in a combination of non-central potentials. This calculation has been done by the path integral solution of the system with/without using Kustannheimo–Stiefel transformation in [9]. Energy eigenfunctions are derived for radial and polar angle parts of the Schrödinger equation with non-central potential, and radial and polar angle wavefunctions are found in

terms of Laguerre and Jacobi polynomials, respectively. When $\alpha = -\eta\sigma^2 e^2$, $Z = \eta\sigma^2$, $\beta = \frac{q\eta^2\sigma^2\hbar^2}{2\mu}$ and $\gamma = 0$, a non-central potential reduces to the Hartmann potential. The energy spectrum of the Hartmann system is obtained as in [2, 7] and polar angle wavefunctions are found in terms of ultraspherical Jacobi polynomials. The relevant results of the Schrödinger equation for not only non-central but also special cases of non-central, i.e., Hartmann and Coulomb, potentials are compared with studies using different methods in the literature.

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